On the Roman bondage number of a graph*

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Abstract

A *Roman dominating function* on a graph $G = (V, E)$ is a function $f : V \to \{0, 1, 2\}$ such that every vertex $v \in V$ with $f(v) = 0$ has at least one neighbor $u \in V$ with $f(u) = 2$. The *weight* of a Roman dominating function is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$. The minimum weight of a Roman dominating function on a graph G is called the *Roman domination number*, denoted by $\gamma_R(G)$. The Roman bondage number $b_R(G)$ of a graph G with maximum degree at least two is the minimum cardinality of all sets $E' \subseteq E(G)$ for which $\gamma_R(G - E') > \gamma_R(G)$. In this paper, we first show that the decision problem for determining $b_{\rm R}(G)$ is NP-hard even for bipartite graphs and then we establish some sharp bounds for $b_{\rm R}(G)$ and characterizes all graphs attaining some of these bounds.

Keywords: Roman domination number, Roman bondage number, NP-hardness.
MSC 2010: 05C69

*The work was supported by NNSF of China (No. 11071233).
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1 Introduction

For terminology and notation on graph theory not given here, the reader is referred to [13, 14, 35]. In this paper, G is a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order $|V|$ of G is denoted by $n = n(G)$. For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V \mid uv \in E\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $\deg_G(v) = \deg(v) = |N(v)|$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. The *open neighborhood* of a set $S \subseteq V$ is the set $N(S) = \cup_{v \in S} N(v)$, and the *closed neighborhood* of S is the set $N[S] = N(S) \cup S$. The *complement* $\overline{G}$ of G is the simple graph whose vertex set is V and whose edges are the pairs of nonadjacent vertices of G. We write K_n for the *complete graph* of order n and C_n for a *cycle* of length n. For two disjoint nonempty sets $S, T \subset V(G)$, $E_G(S, T) = E(S, T)$ denotes the set of edges between S and T.

A subset S of vertices of G is a *dominating set* if $|N(v) \cap S| \geq 1$ for every $v \in V - S$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. To measure the vulnerability or the stability of the domination in an interconnection network under edge failure, Fink et at. [10] proposed the concept of the bondage number in 1990. The *bondage number*, denoted by $b(G)$, of G is the minimum number of edges whose removal from G results in a graph with larger domination number. An edge set B for which $\gamma(G - B) > \gamma(G)$ is called a *bondage set*. A $b(G)$-set is a bondage set of G of size $b(G)$. If B is a $b(G)$-set, then obviously

$$\gamma(G - B) = \gamma(G) + 1. \tag{1}$$

A *Roman dominating function* on a graph G is a labeling $f : V \to \{0, 1, 2\}$ such that every vertex with label 0 has at least one neighbor with label 2. The weight of a Roman dominating function is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$, denoted by $f(G)$. The minimum weight of a Roman dominating function on a graph G is called the *Roman domination number*, denoted by $\gamma_R(G)$. A $\gamma_R(G)$-*function* is a Roman dominating function on G with weight $\gamma_R(G)$. A Roman dominating function $f : V \to \{0, 1, 2\}$ can be represented by the ordered partition (V_0, V_1, V_2) (or (V_0^f, V_1^f, V_2^f) to refer to f) of V, where $V_i = \{v \in V \mid f(v) = i\}$. In this representation, its weight is $\omega(f) = |V_1| + 2|V_2|$. It is clear that $V_1^f \cup V_2^f$ is a dominating set of G, called *the Roman dominating set*, denoted by $D_{\rm R}^f = (V_1, V_2)$. Since $V_1^f \cup V_2^f$ is a dominating set when f is an RDF, and since placing weight 2 at the vertices of a dominating set yields an RDF, in [4], it was observed that

$$\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G). \tag{2}$$

A graph G is called to be *Roman* if $\gamma_{\rm R}(G) = 2\gamma(G)$.

The definition of the Roman dominating function was given implicitly by Stewart [26] and ReVelle and Rosing [25]. Cockayne, Dreyer Jr., Hedetniemi and Hedetniemi [4] as well as Chambers, Kinnersley, Prince and West [3] have given a lot of results on Roman domination. For more information on Roman domination we refer the reader to [3–5, 9, 11, 16–18, 21, 22, 27–30, 33].

Let G be a graph with maximum degree at least two. The *Roman bondage number* $b_R(G)$ of G is the minimum cardinality of all sets $E' \subseteq E$ for which $\gamma_R(G - E') >$

$\gamma_R(G)$. Since in the study of Roman bondage number the assumption $\Delta(G) \geq 2$ is necessary, we always assume that when we discuss $b_R(G)$, all graphs involved satisfy $\Delta(G) \geq 2$. The Roman bondage number $b_R(G)$ was introduced by Jafari Rad and Volkmann in [23], and has been further studied for example in [?, 1, 6–8, 24].

An edge set B that $\gamma_{\rm R}(G-B) > \gamma_{\rm R}(G)$ is called the *Roman bondage set*. A $b_R(G)$-set is a Roman bondage set of G of size $b_R(G)$. If B is a $b_R(G)$-set, then clearly

$$\gamma_{\rm R}(G-B) = \gamma_{\rm R}(G) + 1. \tag{3}$$

In this paper, we first show that the decision problem for determining $b_{\rm R}(G)$ is NP-hard even for bipartite graphs and then we establish some sharp bounds for $b_{\rm R}(G)$ and characterizes all graphs attaining some of these bounds.

We make use of the following results in this paper.

Proposition A. (Chambers et al. [3]) *If G is a graph of order n, then $\gamma_R(G) \leq n - \Delta(G) + 1$.*

Proposition B. (Cockayne et al. [4]) *For a grid graph $P_2 \times P_n$,*

$$\gamma_{\rm R}(P_2 \times P_n) = n + 1.$$

Proposition C. (Cockayne et al. [4]) *For any graph G, $\gamma(G) \leq \gamma_{\rm R}(G) \leq 2\gamma(G)$.*

Proposition D. (Cockayne et al. [4]) *For any graph G of order n, $\gamma(G) = \gamma_{\rm R}(G)$ if and only if $G = \bar{K}_n$.*

Proposition E. (Cockayne et al. [4]) *If G is a connected graph of order n, then $\gamma_{\rm R}(G) = \gamma(G) + 1$ if and only if there is a vertex $v \in V(G)$ of degree $n - \gamma(G)$.*

Proposition F. (Hu and Xu [20]) *If $G = K_{3,3,\ldots,3}$ is the complete t-partite graph of order $n \geq 9$, then $b_{\rm R}(G) = n - 1$.*

Proposition G. (Jafari Rad and Volkmann [23]) *If G is a connected graph of order $n \geq 3$, then $b_{\rm R}(G) \leq \delta(G) + 2\Delta(G) - 3$.*

Proposition H. (Fink et al. [10], Rad and Volkmann [23]) For a cycle C_n of order n,

$$b(C_n) = \begin{cases} 3, & \text{if } n = 1 \,(\text{mod}\, 3); \\ 2, & \text{otherwise.} \end{cases}$$

$$b_{\rm R}(C_n) = \begin{cases} 3, & \text{if } n = 2 \,(\text{mod}\, 3); \\ 2, & \text{otherwise.} \end{cases}$$

Observation 1. *Let G be a connected graph of order $n \geq 3$. Then $\gamma_{\rm R}(G) = 2$ if and only if $\Delta(G) = n - 1$.*

Observation 2. *Let G be a graph of order n with maximum degree at least two. Assume that H is a spanning subgraph of G with $\gamma_{\rm R}(H) = \gamma_{\rm R}(G)$. If $K = E(G) - E(H)$, then $b_{\rm R}(H) \leq b_{\rm R}(G) \leq b_{\rm R}(H) + |K|$.*

Proposition I. *Let G be a nonempty graph of order $n \geq 3$, then $\gamma_{\rm R}(G) = 3$ if and only if $\Delta(G) = n - 2$.*

Proof. Let $\Delta(G) = n-2$. Assume that u is a vertex of degree $n-2$ and v is the unique vertex not adjacent to u in G. By Observation 1, $\gamma_{\rm R}(G) \geq 3$ and clearly $f = (V(G) - \{u, v\}, \{v\}, \{u\})$ is a Roman dominating set of G with $f(G) = 3$. Thus, $\gamma_{\rm R}(G) = 3$.

Conversely, assume $\gamma_{\rm R}(G) = 3$. Then $\Delta(G) \leq n-2$ by Proposition A. Let $f = (V_0, V_1, V_2)$ be a $\gamma_{\rm R}$-function of G. If $V_2 = \emptyset$, then $f(v) = 1$ for each vertex $v \in V(G)$, and hence $n = 3$. Sine G is nonempty and $\Delta(G) \leq n-2 = 1$, we have $\Delta(G) = n-2 = 1$. Let $V_2 \neq \emptyset$. Since $\gamma_{\rm R}(G) = 3$, we deduce that $|V_1| = |V_2| = 1$. Suppose $V_1 = \{v\}$ and $V_2 = \{u\}$. Then other $n-2$ vertices assigned 0 are must be adjacent to u. Thus, $\Delta(G) \geq d_G(u) \geq n-2$ and hence $\Delta(G) = n-2$. $\square$

2 Complexity of Roman bondage number

In this section, we will show that the Roman bondage number problem is NP-hard and the Roman domination number problem is NP-complete even for bipartite graphs. We first state the problem as the following decision problem.

Roman bondage number problem (RBN):

Instance: *A nonempty bipartite graph G and a positive integer k.*

Question: *Is $b_{\rm R}(G) \leq k$?*

Roman domination number problem (RDN):

Instance: *A nonempty bipartite graph G and a positive integer k.*

Question: *Is $\gamma_{\rm R}(G) \leq k$?*

Following Garey and Johnson's techniques for proving NP-completeness given in [12], we prove our results by describing a polynomial transformation from the known-well NP-complete problem: 3SAT. To state 3SAT, we recall some terms.

Let U be a set of Boolean variables. A *truth assignment* for U is a mapping $t: U \to \{T, F\}$. If $t(u) = T$, then u is said to be "true" under t; If $t(u) = F$, then u is said to be "false" under t. If u is a variable in U, then u and $\bar{u}$ are *literals* over U. The literal u is true under t if and only if the variable u is true under t; the literal $\bar{u}$ is true if and only if the variable u is false.

A *clause* over U is a set of literals over U. It represents the disjunction of these literals and is *satisfied* by a truth assignment if and only if at least one of its members is true under that assignment. A collection $\mathscr{C}$ of clauses over U is *satisfiable* if and only if there exists some truth assignment for U that simultaneously satisfies all the clauses in $\mathscr{C}$. Such a truth assignment is called a *satisfying truth assignment* for $\mathscr{C}$. The 3SAT is specified as follows.

3-satisfiability problem (3SAT):

Instance: *A collection* $\mathscr{C} = \{C_1, C_2, \dots, C_m\}$ *of clauses over a finite set* U *of variables such that* $|C_j| = 3$ *for* $j = 1, 2, \dots, m$.

Question: *Is there a truth assignment for* U *that satisfies all the clauses in* $\mathscr{C}$*?*

Theorem 3. (Theorem 3.1 in [12]) *3SAT is NP-complete.*

Theorem 4. *RBN is NP-hard even for bipartite graphs.*

Proof. The transformation is from 3SAT. Let $U = \{u_1, u_2, \dots, u_n\}$ and $\mathscr{C} = \{C_1, C_2, \dots, C_m\}$ be an arbitrary instance of 3SAT. We will construct a bipartite graph G and choose an integer k such that $\mathscr{C}$ is satisfiable if and only if $b_{\rm R}(G) \le k$. We construct such a graph G as follows.

For each $i = 1, 2, \dots, n$, corresponding to the variable $u_i \in U$, associate a graph H_i with vertex set $V(H_i) = \{u_i, \bar{u}_i, v_i, v_i', x_i, y_i, z_i, w_i\}$ and edge set $E(H_i) = \{u_iv_i, u_iz_i, \bar{u}_iv_i', \bar{u}_iz_i, y_iv_i, y_iv_i', y_iz_i, w_iv_i, w_iv_i', w_iz_i, x_iv_i, x_iv_i'\}$. For each $j = 1, 2, \dots, m$, corresponding to the clause $C_j = \{p_j, q_j, r_j\} \in \mathscr{C}$, associate a single vertex c_j and add edge set $E_j = \{c_jp_j, c_jq_j, c_jr_j\}$, $1 \le j \le m$. Finally, add a path $P = s_1s_2s_3$, join s_1 and s_3 to each vertex c_j with $1 \le j \le m$ and set $k = 1$.

Figure 1 shows an example of the graph obtained when $U = \{u_1, u_2, u_3, u_4\}$ and $\mathscr{C} = \{C_1, C_2, C_3\}$, where $C_1 = \{u_1, u_2, \bar{u}_3\}, C_2 = \{\bar{u}_1, u_2, u_4\}, C_3 = \{\bar{u}_2, u_3, u_4\}$.

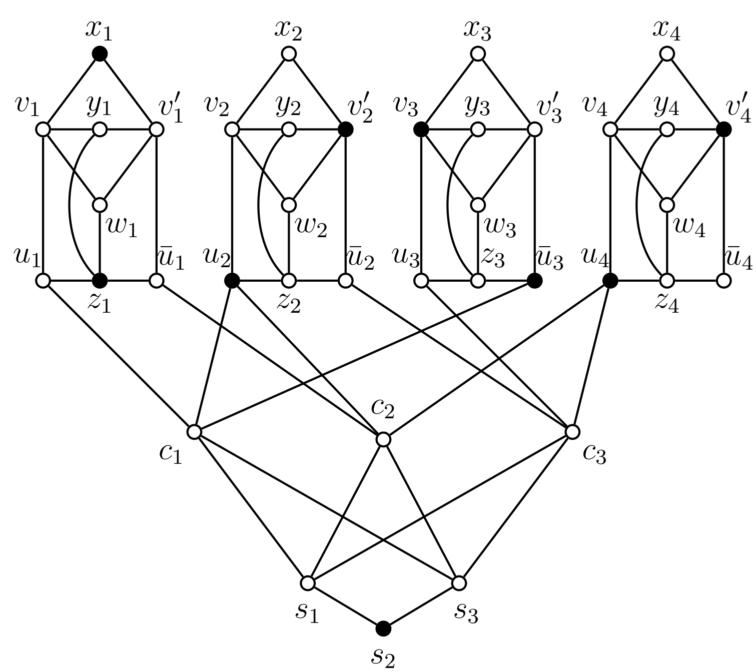

Figure 1: An instance of the Roman bondage number problem resulting from an instance of 3SAT. Here $k = 1$ and $\gamma_{\rm R}(G) = 18$, where the bold vertex w means a Roman dominating function with $f(w) = 2$.

To prove that this is indeed a transformation, we only need to show that $b_{\mathrm{R}}(G)=1$ if and only if there is a truth assignment for U that satisfies all clauses in $\mathscr{C}$. This aim can be obtained by proving the following four claims.

Claim 4.1 $\gamma_{\mathrm{R}}(G)\geq 4n+2$. *Moreover, if* $\gamma_{\mathrm{R}}(G)=4n+2$, *then for any* γ_{R}*-function* f *on* G, $f(H_i)=4$ *and at most one of* $f(u_i)$ *and* $f(\bar{u}_i)$ *is 2 for each* i, $f(c_j)=0$ *for each* j *and* $f(s_2)=2$.

Proof. Let f be a γ_{R}-function of G, and let $H_i'=H_i-u_i-\bar{u}_i$.

If $f(u_i)=2$ and $f(\bar{u}_i)=2$, then $f(H_i)\geq 4$. Assume either $f(u_i)=2$ or $f(\bar{u}_i)=2$, if $f(x_i)=0$ or $f(y_i)=0$, then there is at least one vertex t in $\{v_i,v_i',z_i\}$ such that $f(t)=2$. And hence $f(H_i')\geq 2$. Thus, $f(H_i)\geq 4$.

If $f(u_i)\neq 2$ and $f(\bar{u}_i)\neq 2$, let f' be a restriction of f on H_i', then f' is a Roman dominating function of H_i', and $f'(H_i')\geq\gamma_{\mathrm{R}}(H_i')$. Since the maximum degree of H_i' is $V(H_i')-3$, by Lemma I, $\gamma_{\mathrm{R}}(H_i')>3$ and hence $f'(H_i')\geq 4$ and $f(H_i)\geq 4$. If $f(s_1)=0$ or $f(s_3)=0$, then there is at least one vertex t in $\{c_1,\cdots,c_m,s_2\}$ such that $f(t)=2$. Then $f(N_G[V(P)])\geq 2$, and hence $\gamma_{\mathrm{R}}(G)\geq 4n+2$.

Suppose that $\gamma_{\mathrm{R}}(G)=4n+2$, then $f(H_i)=4$ and since $f(N_G[x_i])\geq 1$, at most one of $f(u_i)$ and $f(\bar{u}_i)$ is 2 for each $i=1,2,\ldots,n$, while $f(N_G[V(P)])=2$. It follows that $f(s_2)=2$ since $f(N_G[s_2])\geq 1$. Consequently, $f(c_j)=0$ for each $j=1,2,\ldots,m$. $\square$

Claim 4.2 $\gamma_{\mathrm{R}}(G)=4n+2$ *if and only if* $\mathscr{C}$ *is satisfiable.*

Proof. Suppose that $\gamma_{\mathrm{R}}(G)=4n+2$ and let f be a γ_{R}-function of G. By Claim 4.1, at most one of $f(u_i)$ and $f(\bar{u}_i)$ is 2 for each $i=1,2,\ldots,n$. Define a mapping $t:U\to\{T,F\}$ by

$$t(u_i)=\begin{cases} T & \text{if } f(u_i)=2 \text{ or } f(u_i)\neq 2 \text{ and} f(\bar{u}_i)\neq 2,\\ F & \text{if } f(\bar{u}_i)=2.\end{cases}\qquad i=1,2,\ldots,n. \tag{4}$$

We now show that t is a satisfying truth assignment for $\mathscr{C}$. It is sufficient to show that every clause in $\mathscr{C}$ is satisfied by t. To this end, we arbitrarily choose a clause $C_j\in\mathscr{C}$ with $1\leq j\leq m$.

By Claim 4.1, $f(c_j)=f(s_1)=f(s_3)=0$. There exists some i with $1\leq i\leq n$ such that $f(u_i)=2$ or $f(\bar{u}_i)=2$ where c_j is adjacent to u_i or $\bar{u}_i$. Suppose that c_j is adjacent to u_i where $f(u_i)=2$. Since u_i is adjacent to c_j in G, the literal u_i is in the clause C_j by the construction of G. Since $f(u_i)=2$, it follows that $t(u_i)=T$ by (4), which implies that the clause C_j is satisfied by t. Suppose that c_j is adjacent to $\bar{u}_i$ where $f(\bar{u}_i)=2$. Since $\bar{u}_i$ is adjacent to c_j in G, the literal $\bar{u}_i$ is in the clause C_j. Since $f(\bar{u}_i)=2$, it follows that $t(u_i)=F$ by (4). Thus, t assigns $\bar{u}_i$ the truth value T, that is, t satisfies the clause C_j. By the arbitrariness of j with $1\leq j\leq m$, we show that t satisfies all the clauses in $\mathscr{C}$, that is, $\mathscr{C}$ is satisfiable.

Conversely, suppose that $\mathscr{C}$ is satisfiable, and let $t:U\to\{T,F\}$ be a satisfying truth assignment for $\mathscr{C}$. Create a function f on $V(G)$ as follows: if $t(u_i)=T$,

then let $f(u_i) = f(v_i') = 2$, and if $t(u_i) = F$, then let $f(\bar{u}_i) = f(v_i) = 2$. Let $f(s_2) = 2$. Clearly, $f(G) = 4n + 2$. Since t is a satisfying truth assignment for $\mathscr{C}$, for each $j = 1, 2, \ldots, m$, at least one of literals in C_j is true under the assignment t. It follows that the corresponding vertex c_j in G is adjacent to at least one vertex w with $f(w) = 2$ since c_j is adjacent to each literal in C_j by the construction of G. Thus f is a Roman dominating function of G, and so $\gamma_{\rm R}(G) \leq f(G) = 4n + 2$. By Claim 4.1, $\gamma_{\rm R}(G) \geq 4n + 2$, and so $\gamma_{\rm R}(G) = 4n + 2$. □

Claim 4.3 $\gamma_{\rm R}(G - e) \leq 4n + 3$ *for any* $e \in E(G)$.

Proof. For any edge $e \in E(G)$, it is sufficient to construct a Roman dominating function f on $G-e$ with weight $4n+3$. We first assume $e \in E_G(s_1)$ or $e \in E_G(s_3)$ or $e \in E_G(c_j)$ for some $j = 1, 2, \ldots, m$, without loss of generality let $e \in E_G(s_1)$ or $e = c_j u_i$ or $e = c_j \bar{u}_i$. Let $f(s_3) = 2, f(s_1) = 1$ and $f(u_i) = f(v_i') = 2$ for each $i = 1, 2, \ldots, n$. For the edge $e \notin E_G(u_i)$ and $e \notin E_G(v_i')$, let $f(s_1) = 2, f(s_3) = 1$ and $f(u_i) = f(v_i') = 2$. For the edge $e \notin E(\bar{u}_i)$ and $e \notin E(v_i)$, let $f(s_1) = 2, f(s_3) = 1$ and $f(\bar{u}_i) = f(v_i) = 2$. If $e = u_i v_i$ or $e = \bar{u}_i v_i'$, let $f(s_1) = 2, f(s_3) = 1$ and $f(x_i) = f(z_i) = 2$. Then f is a Roman dominating function of $G - e$ with $f(G - e) = 4n + 3$ and hence $\gamma_{\rm R}(G - e) \leq 4n + 3$. □

Claim 4.4 $\gamma_{\rm R}(G) = 4n + 2$ *if and only if* $b_{\rm R}(G) = 1$.

Proof. Assume $\gamma_{\rm R}(G) = 4n + 2$ and consider the edge $e = s_1 s_2$. Suppose $\gamma_{\rm R}(G) = \gamma_{\rm R}(G - e)$. Let f' be a $\gamma_{\rm R}$-function of $G - e$. It is clear that f' is also a $\gamma_{\rm R}$-function on G. By Claim 4.1 we have $f'(c_j) = 0$ for each $j = 1, 2, \ldots, m$ and $f'(s_2) = 2$. But then $f'(N_{G-e}[s_1]) = 0$, a contradiction. Hence, $\gamma_{\rm R}(G) < \gamma_{\rm R}(G - e)$, and so $b_{\rm R}(G) = 1$.

Now, assume $b_{\rm R}(G) = 1$. By Claim 4.1, we have $\gamma_{\rm R}(G) \geq 4n + 2$. Let e' be an edge such that $\gamma_{\rm R}(G) < \gamma_{\rm R}(G - e')$. By Claim 4.3, we have that $\gamma_{\rm R}(G - e') \leq 4n + 3$. Thus, $4n + 2 \leq \gamma_{\rm R}(G) < \gamma_{\rm R}(G - e') \leq 4n + 3$, which yields $\gamma_{\rm R}(G) = 4n + 2$. □

By Claim 4.2 and Claim 4.4, we prove that $b_{\rm R}(G) = 1$ if and only if there is a truth assignment for U that satisfies all clauses in $\mathscr{C}$. Since the construction of the Roman bondage number instance is straightforward from a 3-satisfiability instance, the size of the Roman bondage number instance is bounded above by a polynomial function of the size of 3-satisfiability instance. It follows that this is a polynomial reduction and the proof is complete. □

Corollary 5. *Roman domination number problem is NP-complete even for bipartite graphs.*

Proof. It is easy to see that the Roman domination problem is in NP since a nondeterministic algorithm need only guess a vertex set pair (V_1, V_2) with $|V_1| + 2|V_2| \leq k$ and check in polynomial time whether that for any vertex $u \in V \setminus (V_1 \cup V_2)$ whether there is a vertex in V_2 adjacent to u for a given nonempty graph G.

We use the same method as Theorem 4 to prove this conclusion. We construct the same graph G but does not contain the path P. We set $k = 4n$, then use the same methods as Claim 4.1 and 4.2, we have that $\gamma_{\mathrm{R}}(G) = 4n$ if and only if $\mathscr{C}$ is satisfiable. $\square$

3 General bounds

Lemma 6. *Let G be a connected graph of order $n \geq 3$ such that $\gamma_{\mathrm{R}}(G) = \gamma(G) + 1$. If there is a set B of edges with $\gamma_{\mathrm{R}}(G - B) = \gamma_{\mathrm{R}}(G)$, then $\Delta(G) = \Delta(G - B)$.*

Proof. Since G is connected and $n \geq 3$, $\gamma_{\mathrm{R}}(G) = \gamma(G)+1 \leq n-1$. Since $\gamma_{\mathrm{R}}(G-B) = \gamma_{\mathrm{R}}(G) \leq n-1$, $G - B$ is nonempty. It follows from Propositions C and D that $\gamma_{\mathrm{R}}(G - B) \geq \gamma(G - B) + 1$. Since

$$\gamma_{\mathrm{R}}(G - B) = \gamma_{\mathrm{R}}(G) = \gamma(G) + 1 \leq \gamma(G - B) + 1,$$

we have $\gamma_{\mathrm{R}}(G - B) = \gamma(G - B) + 1$, and then $\gamma(G - B) = \gamma(G)$.

If $G - B$ is connected, then by Proposition E,

$$\Delta(G - B) = n - \gamma(G - B) = n - \gamma(G) = \Delta(G).$$

If $G - B$ is disconnected, then let G_1 be a nonempty connected component of $G - B$. By Propositions C and D, $\gamma_{\mathrm{R}}(G_1) \geq \gamma(G_1) + 1$. Then

$$\begin{aligned} \gamma(G) + 1 &= \gamma_{\mathrm{R}}(G - B) \\ &= \gamma_{\mathrm{R}}(G_1) + \gamma_{\mathrm{R}}(G - G_1) \\ &\geq \gamma(G_1) + 1 + \gamma(G - G_1) \\ &\geq \gamma(G) + 1, \end{aligned}$$

and hence $\gamma_{\mathrm{R}}(G_1) = \gamma(G_1)+1$, $\gamma_{\mathrm{R}}(G-G_1) = \gamma(G-G_1)$ and $\gamma(G) = \gamma(G_1)+\gamma(G-G_1)$. By Proposition D, $G - G_1$ is empty and hence $\gamma(G - G_1) = |V(G - G_1)|$. By Proposition E,

$$\begin{aligned} \Delta(G_1) &= |V(G_1)| - \gamma(G_1) \\ &= n - |V(G - G_1)| - \gamma(G_1) \\ &= n - \gamma(G - G_1) - \gamma(G_1) \\ &= n - \gamma(G) = \Delta(G) \end{aligned}$$

as desirable. $\square$

Theorem 7. *Let G be a connected graph of order $n \geq 3$ with $\gamma_{\mathrm{R}}(G) = \gamma(G) + 1$. Then*

$$b_{\mathrm{R}}(G) \leq \min\{b(G), n_\Delta\},$$

where n_Δ is the number of vertices with maximum degree Δ in G.

Proof. Since $n \geq 3$ and G is connected, we have $\Delta(G) \geq 2$ and hence $\gamma(G) \leq n - 2$. Let B be a $b(G)$- set. By (1), $\gamma(G-B) = \gamma(G)+1 \leq n-1$ and so $G-B$ is nonempty. It follows from Propositions C and D that $\gamma_{\mathrm{R}}(G-B) \geq \gamma(G-B)+1 > \gamma(G)+1 = \gamma_{\mathrm{R}}(G)$ and hence B is a Roman bondage set of G. Thus, $b_{\mathrm{R}}(G) \leq b(G)$.

We now prove that $b_{\rm R}(G) \leq n_\Delta$. It follows from Propositions A, E and the fact $\gamma_{\rm R}(G) = \gamma(G) + 1$ that $\Delta(G) = n - \gamma(G)$. Let $\{v_1, \ldots, v_{n_\Delta}\}$ be the set consists of all vertices of degree Δ and let e_i be an edge adjacent to v_i for each $1 \leq i \leq n_\Delta$. Suppose $B' = \{e_1, \ldots, e_{n_\Delta}\}$. Clearly, $\Delta(G - B') < \Delta(G) = n - \gamma(G)$ and $G - B'$ is nonempty. Since $G - B'$ is nonempty, it follows from Propositions C and D that $\gamma_{\rm R}(G - B') \geq \gamma(G - B') + 1$. We claim that $\gamma_{\rm R}(G - B') > \gamma_{\rm R}(G)$. Assume to the contrary that $\gamma_{\rm R}(G - B') = \gamma_{\rm R}(G)$. We deduce from Lemma 6 that $\Delta(G - B') = \Delta(G) = n - \gamma(G)$, a contradiction. Hence $b_{\rm R}(G) \leq |B'| \leq n_\Delta$. This completes the proof. $\square$

Theorem 8. *For every Roman graph G,*

$$b_{\rm R}(G) \geq b(G).$$

The bound is sharp for cycles on n vertices where $n \equiv 0 \pmod 3$.

Proof. Let B be a $b_R(G)$- set. Then by (2) we have

$$2\gamma(G - B) \geq \gamma_{\rm R}(G - B) > \gamma_{\rm R}(G) = 2\gamma(G).$$

Thus $\gamma(G - B) > \gamma(G)$ and hence $b_{\rm R}(G) \geq b(G)$.

By Proposition H, we have $b_{\rm R}(C_n) \geq b(C_n) = 2$ when $n \equiv 0 \pmod 3$. $\square$

The strict inequality in Theorem 8 can hold, for example, $b(C_{3k+2}) = 2 < 3 = b_{\rm R}(C_{3k+2})$ by Proposition H.

A graph G is called to be *vertex domination-critical* (*vc-graph* for short) if $\gamma(G - x) < \gamma(G)$ for any vertex x in G. We call a graph G to be *vertex Roman domination-critical* (*vrc-graph* for short) if $\gamma_{\rm R}(G - x) < \gamma_{\rm R}(G)$ for every vertex x in G.

The *vertex covering number* $\beta(G)$ of G is the minimum number of vertices that are incident with all edges in G. If G has no isolated vertices, then $\gamma_{\rm R}(G) \leq 2\gamma(G) \leq 2\beta(G)$. If $\gamma_{\rm R}(G) = 2\beta(G)$, then $\gamma_{\rm R}(G) = 2\gamma(G)$ and hence G is a Roman graph. In [31], Volkmann gave a lot of graphs with $\gamma(G) = \beta(G)$.

Theorem 9. *Let G be a graph with $\gamma_{\rm R}(G) = 2\beta(G)$. Then*
(1) $b_{\rm R}(G) \geq \delta(G)$;
(2) $b_{\rm R}(G) \geq \delta(G) + 1$ if G is a vrc-graph.

Proof. Let G be a graph such that $\gamma_{\rm R}(G) = 2\beta(G)$.

(1) If $\delta(G) = 1$, then the result is immediate. Assume $\delta(G) \geq 2$. Let $B \subseteq E(G)$ and $|B| \leq \delta(G) - 1$. Then $\delta(G - B) \geq 1$ and so $\gamma_{\rm R}(G) \leq \gamma_{\rm R}(G - B) \leq 2\beta(G - B) \leq 2\beta(G) = \gamma_{\rm R}(G)$. Thus, B is not a Roman bondage set of G, and hence $b_{\rm R}(G) \geq \delta(G)$.

(2)Let B be a Roman bondage set of G. An argument similar to that described in the proof of (1), shows that B must contain all edges incident with some vertex of G, say x. Hence, $G - B$ has an isolated vertex. On the other hand, since G is a vrc-graph, $\gamma_{\rm R}(G - x) < \gamma_{\rm R}(G)$ which implies that the removal of all edges incident to x can not increase the Roman domination number. Hence, $b_{\rm R}(G) \geq \delta(G) + 1$. $\square$

The *cartesian product* $G = G_1 \times G_2$ of two disjoint graphs G_1 and G_2 has $V(G) = V(G_1) \times V(G_2)$, and two vertices (u_1, u_2) and (v_1, v_2) of G are adjacent if and only if either $u_1 = v_1$ and $u_2v_2 \in E(G_2)$ or $u_2 = v_2$ and $u_1v_1 \in E(G_1)$. The cartesian product of two paths $P_r = x_1x_2 \ldots x_r$ and $P_t = y_1y_2 \ldots y_t$ is called a *grid*. Let $G_{r,s} = P_r \times P_t$ is a grid, and let $V(G_{r,s}) = \{u_{i,j} = (x_i, y_j) | 1 \leq i \leq r \text{ and } 1 \leq j \leq t\}$ be the vertex set of G. Next we determine Roman bondage number of grids.

Theorem 10. *For $n \geq 2$, $b_{\rm R}(G_{2,n}) = 2$.*

Proof. By Proposition B, we have $\gamma_{\rm R}(G_{2,n}) = n+1$. Since

$$\gamma_{\rm R}(G_{2,n} - u_{1,1}u_{1,2} - u_{2,1}u_{2,2}) = 2 + \gamma_{\rm R}(G_{2,n-1}) = n+2,$$

we deduce that $b_{\rm R}(G_{2,n}) \leq 2$. Now we show that $\gamma_{\rm R}(G_{2,n} - e) = \gamma_{\rm R}(G_{2,n})$ for any edge $e \in E(G_{2,n})$. Consider two cases.

Case 1 n is odd.
For $i = 1, 2, 3, 4$, define $f_i : V(G_{2,n}) \to \{0, 1, 2\}$ as follows:

$$f_1(u_{i,j}) = \begin{cases} 2 & \text{if} \quad i = 1 \text{ and } j \equiv 1 \pmod 4 \ \text{ or } \ i = 2 \text{ and } j \equiv 3 \pmod 4 \\ 0 & \text{if} \quad \text{otherwise,} \end{cases}$$

$$f_2(u_{i,j}) = \begin{cases} 2 & \text{if} \quad i = 1 \text{ and } j \equiv 3 \pmod 4 \ \text{ or } \ i = 2 \text{ and } j \equiv 1 \pmod 4 \\ 0 & \text{if} \quad \text{otherwise,} \end{cases}$$

and if $n \equiv 1 \pmod 4$, then

$$f_3(u_{i,j}) = \begin{cases} 2 & \text{if} \quad i = 1 \text{ and } j \equiv 0 \pmod 4 \ \text{ or } \ i = 2 \text{ and } j \equiv 2 \pmod 4 \\ 1 & \text{if} \quad i = j = 1 \ \text{ or } \ i = 2 \text{ and } j = n \\ 0 & \text{if} \quad \text{otherwise.} \end{cases}$$

and if $n \equiv 3 \pmod 4$, then

$$f_4(u_{i,j}) = \begin{cases} 2 & \text{if} \quad i = 1 \text{ and } j \equiv 2 \pmod 4 \ \text{ or } \ i = 2 \text{ and } j \equiv 0 \pmod 4 \\ 1 & \text{if} \quad i = 2 \text{ and } j = 1 \ \text{ or } \ i = 2 \text{ and } j = n \\ 0 & \text{if} \quad \text{otherwise.} \end{cases}$$

Obviously, f_i is a $\gamma_R(G_{2,n})$-function for each $i = 1, 2, 3$ when $n \equiv 1 \pmod 4$ and f_i is a $\gamma_R(G_{2,n})$-function for each $i = 1, 2, 4$ when $n \equiv 3 \pmod 4$. Let $e \in E(G)$ be an arbitrary edge of G. Then clearly , f_1 or f_2 or f_3 is a Roman dominating function of $G - e$ if $n \equiv 1 \pmod 4$ and f_1 or f_2 or f_3 is a Roman dominating function of $G - e$ if $n \equiv 3 \pmod 4$. Hence $b_R(G_{2,n}) \geq 2$.

Case 2 n is even.
For $i = 1, 2, 3, 4$, define $f_i : V(G_{2,n}) \to \{0, 1, 2\}$ as follows:

$$f_1(u_{i,j}) = \begin{cases} 2 & \text{if} \quad i = 1 \text{ and } j \equiv 0 \pmod 4 \ \text{ or } \ i = 2 \text{ and } j \equiv 2 \pmod 4 \\ 1 & \text{if} \quad i = j = 1 \\ 0 & \text{if} \quad \text{otherwise,} \end{cases}$$

$$f_2(u_{i,j}) = \begin{cases} 2 & \text{if} \quad i = 1 \text{ and } j \equiv 2 \pmod 4 \ \text{ or } \ i = 2 \text{ and } j \equiv 0 \pmod 4 \\ 1 & \text{if} \quad i = 2 \text{ and } j = 1 \\ 0 & \text{if} \quad \text{otherwise.} \end{cases}$$

and if $n \equiv 0 \pmod 4$, then

$$f_3(u_{i,j}) = \begin{cases} 2 & \text{if} \quad i=1 \text{ and } j \equiv 1 \pmod 4 \quad \text{or} \quad i=2 \text{ and } j \equiv 3 \pmod 4 \\ 1 & \text{if} \quad i=1 \text{ and } j=n \\ 0 & \text{if} \quad \text{otherwise,} \end{cases}$$

and if $n \equiv 2 \pmod 4$, then

$$f_4(u_{i,j}) = \begin{cases} 2 & \text{if} \quad i=1 \text{ and } j \equiv 1 \pmod 4 \quad \text{or} \quad i=2 \text{ and } j \equiv 3 \pmod 4 \\ 1 & \text{if} \quad i=2 \text{ and } j=n \\ 0 & \text{if} \quad \text{otherwise,} \end{cases}$$

Obviously, f_i is a $\gamma_R(G_{2,n})$-function for each $i = 1,2,3$ when $n \equiv 0 \pmod 4$ and f_i is a $\gamma_R(G_{2,n})$-function for each $i = 1,2,4$ when $n \equiv 2 \pmod 4$. Let $e \in E(G)$ be an arbitrary edge of G. Then clearly , f_1 or f_2 or f_3 is a Roman dominating function of $G-e$ if $n \equiv 0 \pmod 4$ and f_1 or f_2 or f_4 is a Roman dominating function of $G-e$ if $n \equiv 2 \pmod 4$. Hence $b_R(G_{2,n}) \geq 2$. This completes the proof. □

4 Roman bondage number of graphs with small Roman domination number

Dehgardi, Sheikholeslami and Volkmann [7] posed the following problem: If G is a connected graph of order $n \geq 4$ with Roman domination number $\gamma_R(G) \geq 3$, then

$$b_R(G) \leq (\gamma_R(G) - 2)\Delta(G). \tag{5}$$

Theorem I shows that the inequality (5) holds if $\gamma_R(G) \geq 5$. Thus the bound in (5) is of interest only when $\gamma_R(G)$ is 3 or 4. In this section we prove (5) for all graphs G of order $n \geq 4$ with $\gamma_R(G) = 3, 4$, improving Proposition I.

Theorem 11. *If G is a connected graph of order $n \geq 4$ with $\gamma_R(G) = 3$, then*

$$b_R(G) \leq \Delta(G) = n - 2.$$

Proof. Let $\gamma_R(G) = 3$. Then $\Delta(G) = n-2$ by Proposition I. Let M be maximum matching of G and let U be the set consisting of unsaturated vertices. Since G is connected and $\gamma_R(G) = 3$, we deduce that $|M| \geq 2$.

If $U = \emptyset$, then $G - M$ has no vertex of degree $n-2$ and it follows from Proposition I that $\gamma_R(G - M) \geq 4$. Thus

$$b_R(G) \leq |M| \leq \frac{n}{2} \leq n - 2 = \Delta(G). \tag{6}$$

Assume now that $U \neq \emptyset$. Clearly U is an independent set. Since G is connected and M is maximum, there exist a set J of $|U|$ edges such that each vertex of U is incident with exactly one edge of J. Then $|J| = |U| = n - 2|M|$. Now let $F = J \cup M$. Obviously, $G - F$ has no vertex of degree $n-2$, and it follows from Proposition I that $\gamma_R(G_F) \geq 4$. This implies that

$$b_R(G) \leq |M| + |U| = n - |M| \leq n - 2 = \Delta(G). \tag{7}$$

This completes the proof. □

Next we characterize all graphs that achieve the bound in Theorem 11.

Theorem 12. *If equality holds in Theorem 11, then G is regular.*

Proof. Let $\gamma_R(G) = 3$ and $b_R(G) = \Delta(G) = n - 2$. If G has a perfect matching M, then it follows from (6) that $\frac{n}{2} = n - 2$ and hence $n = 4$. This implies that $b_R(G) = |M| = 2 = \Delta(G)$. Since $b_R(P_4) = 1$, we have $G = C_4$ as desired.

Let G does not have a perfect matching and let M be a maximum matching of G. It follows from (7) that $|M| = 2$. Let X be the independent set of M-unsaturated vertices. We consider two cases.

Case 1. $|X| = 1$.
Then $n = 5$. Let $V(G) = \{v_1, \ldots, v_5\}$. Since $\gamma_R(G) = 3$, $\Delta(G) = n - 2 = 3$ by Proposition I. Since n is odd, G has a vertex of even degree 2. Let $\deg(v_1) = 2$ and let $v_1v_2, v_1v_3 \in E(G)$. Since $b_R(G) = 3 > \deg(v_1)$, we have $\gamma_R(G - v_1) = \gamma_R(G) - 1 = 2$. By Observation 1, $\Delta(G - v_1) = 3$. Since $\gamma_R(G) = 3$, we may assume without loss of generality that $\deg(v_4) = 3$ and $\{v_4v_2, v_4v_3, v_4v_5\} \subseteq E(G)$. Let $F = \{v_1v_2, v_3v_4\}$. Since $b_R(G) = 3 > |F|$, we have $\gamma_R(G-F) = 3$. It follows from Proposition I and the fact $\gamma_R(G-F) = 3$ that $\deg_{G-F}(v_5) = 3$. This implies that $\{v_5v_2, v_5v_3, v_5v_4\} \subseteq E(G)$. Thus $E(G) = \{v_1v_2, v_1v_3, v_2v_4, v_2v_5, v_3v_4, v_3v_5, v_4v_5\}$. Now we have $G - \{v_2v_4, v_3v_5\} \simeq C_5$ and hence $\gamma_R(G - \{v_2v_4, v_3v_5\}) = 4$. This implies that $b_R(G) \leq 2$ a contradiction.

Case 2. $|X| \geq 2$.
Then $n \geq 6$. Let $M = \{u_1v_1, u_2v_2\}$ be a maximum matching of G. If y and z are vertices of X and $yu_i \in E(G)$, then since the matching M is maximum, $zv_i \notin E(G)$. Therefore, we may assume without loss of generality that $N_G(X) \subseteq \{u_1, u_2\}$. So $\deg(y) + \deg(z) \leq 4$ for every pair of distinct vertices y and z in X. Let $y, z \in X$ and F be the set of edges incident with y or z. Then y, z are isolated vertices in $G - F$ and hence $\gamma_R(G - F) \geq 4$. If $|F| \leq 3$, then $n - 2 = b_R(G) \leq 3$ which leads to a contradiction. Therefore, $|F| = 4$. It follows that $n-2 = b_R(G) \leq 4$ and hence $n = 6$. Let $V(G) = \{u_1, u_2, v_1, v_2, y, z\}$. Then $\deg(y) = \deg(z) = 2$ and $\deg(u_1), \deg(u_2) \geq 3$. If $v_1v_2 \in E(G)$, then $\{yu_1, xu_2, v_1v_2\}$ is a matching of G which is a contradiction. Thus $\deg(v_1), \deg(v_2) \leq 2$. Since $\gamma_R(G) = 3$, $\Delta(G) = n - 2 = 4$ by Proposition I. We distinguish two subcases.

Subcase 2.1 $\delta(G) = 1$.
Assume without loss of generality that $\deg(v_1) = 1$. Let F be the set of edges incident with y or v_1. Then $|F| = 3$ and y, v_1 are isolated vertices in $G - F$ and hence $\gamma_R(G - F) \geq 4$. Thus $n - 2 = b_R(G) \leq 3$, a contradiction.

Subcase 2.2 $\delta(G) = 2$.
Then we must have $\deg(v_1) = \deg(v_2) = 2$ and $v_1u_2, v_2u_1 \in E(G)$. Let $F = \{yu_1, zu_2\}$. Clearly $\Delta(G - F) = 3 = n - 3$ and it follows from Proposition I that $\gamma_R(G - F) \geq 4$. Hence $b_R(G) \leq 2$, which is a contradiction.

This completes the proof. □

Proposition J. *The complete graph K_{2r} is 1-factorable.*

According to Theorem 11, Theorem 12, Proposition I and Proposition J, we prove the next result.

Theorem 13. *Let G be a connected graph of order $n \geq 4$ with $\gamma_R(G) = 3$. Then $b_R(G) = \Delta(G) = n - 2$ if and only if $G \simeq C_4$.*

Proof. Let G be a connected graph of order $n \geq 4$ with $\gamma_R(G) = 3$. It follows from Theorem 11 that $b_R(G) \leq n - 2$.

If $G \simeq C_4$, then obviously $b_R(G) = 2 = n - 2$.

Conversely, assume that $b_R(G) = n-2$. It follows from Proposition I and Theorem 12 that G is $(n-2)$-regular. This implies that n is even and hence $G = K_n - M$ where M is a perfect matching in K_n. By Proposition J, G is 1-factorable. Let M_1 be a perfect matching in G. Now $G - M_1$ is an $(n-3)$-regular and it follows from Proposition I that $\gamma_R(G - M_1) \geq 4$. Thus $n - 2 = b_R(G) \leq \frac{n}{2}$ which implies that $n = 4$ and hence $G = C_4$. □

Theorem 14. *If G is a connected graph of order $n \geq 4$ with $\gamma_R(G) = 4$, then*

$$b_R(G) \leq \Delta(G) + \delta(G) - 1.$$

Proof. Obviously $\Delta(G) \geq 2$. Let u be a vertex of minimum degree $\delta(G)$. If $b_R(G) \leq \deg(u)$, then we are done. Suppose $b_R(G) > \deg(u)$. Then $\gamma_R(G-u) = \gamma_R(G)-1 = 3$. By Theorem 11, $b_R(G-u) \leq \Delta(G-u)$. If $b_R(G-u) = \Delta(G-u)$, then $G - u = C_4$ by Theorem 13 and since G is connected, we deduce that $\gamma_R(G) = 3$, a contradiction. Thus $b_R(G-u) \leq \Delta(G-u) - 1$. It follows from Observation 2 that

$$b_R(G) \leq b_R(G-u) + \deg(u) \leq \Delta(G-u) - 1 + \deg(u) \leq \Delta(G) + \delta(G) - 1, \quad (8)$$

as desired. This completes the proof. □

Dehgardi et al. [7] proved that for any connected graph G of order $n \geq 3$, $b_{\mathrm{R}}(G) \leq n - 1$ and posed the following problems.

Problem 1. *Prove or disprove: For any connected graph G of order $n \geq 3$, $b_{\mathrm{R}}(G) = n - 1$ if and only if $G \cong K_3$.*

Problem 2. *Prove or disprove: If G is a connected graph of order $n \geq 3$, then*

$$b_{\mathrm{R}}(G) \leq n - \gamma_{\mathrm{R}}(G) + 1.$$

Since $\gamma_R(K_{3,3,\ldots,3}) = 4$, Proposition F shows that Problems 1 and 2 are false. Recently Akbari and Qajar [1] proved that:

Proposition K. *If G is a connected graph of order $n \geq 3$, then*

$$b_R(G) \leq n - \gamma_R(G) + 5.$$

We conclude this paper with the following revised problems.

Problem 3. *Characterize all connected graphs G of order $n \geq 3$ for which $b_R(G) = n - 1$.*

Problem 4. *Prove or disprove: If G is a connected graph of order $n \geq 3$, then*

$$b_R(G) \leq n - \gamma_R(G) + 3.$$

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